Degraded Multicasting with Network Coding over the Combination Network

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***Abstract*— In this paper, we give a characterization of the rate region for the degraded two message set problem, applied to a combination network with erasure channels. We also provide an algorithm that uses topological information in order to deliver the two messages to the receivers, and we show that our algorithm is optimal, in the sense that it achieves any rate pair in the region. We compare our algorithm analytically with a naive approach oblivious to the network structure, and we give an insight on what benefits should be expected for different classes of networks.**

I. Introduction

Content delivery, i.e. multicasting, is an application where network coding promises to have impact, as significant benefits have been observed both theoretically as well as in practice. The case where all receivers require the exact same content is by now well understood; however, for the (perhaps more realistic) case, where different users require different subsets of the content, although there exist a number of proposed heuristic algorithms, there is in general no exact characterization of the optimal achievable rate region [1].

In this paper, we provide such a characterization for the degraded two-message set problem, where a source broadcasts two messages to a set of receivers over a combination network with erasure channels. Degraded broadcasting refers to that the "weaker" receivers receive a subset of the information that the "stronger" users collect. That is, the weaker users require a message W_1, transmitted at a rate R_1, while the stronger users require not only W_1, but also a second message W_2, transmitted at a rate R_2.

Degraded broadcasting is motivated by various scenarios, such as video streaming applications, or broadcasting in the presence of fading. In the first case, users are heterogeneous and have different subscription levels, thus requiring a different resolution of the content [9]. In the second case, the receivers are not able to receive the whole content due to channel fading, that can be modeled as erasures at higher layers.

The problem we solve is a special case of a long-standing open question in multi-user information theory, of delivering a set of degraded messages over a general broadcast channel introduced in [2]. Although special cases have been addressed [3], [4], [5], there is comparatively little understanding when there are more than two users. Recent progress on a particular case of this question has been made in [7]. In [11] the authors introduce the network sharing bound, for a more general setting, however without considering erasures in the network. Closer to our work is the one in [8] that examines two-message broadcasting over a linear deterministic channel; our work differs in that we specifically look at the combination network, incorporate erasures, and provide a simpler achievability scheme.

Our main contributions in this paper are:

- We provide an exact characterization of the rate region for the two-degraded message-set problem, over the combination network and with three receivers.
- We present a very simple achievability scheme, that assigns source messages (or their linear combinations) to the network edges in polynomial time. A main observation from our work is that, to achieve the optimal rates, we need to take into account topological information, namely, what subset of receivers observes each edge.
- We provide an analytical comparison with an approach that is oblivious to the topology, and highlight what are the network topologies where the optimal approach can offer benefits.

A side result of our work is that, to achieve the optimal performance, we only need to use very simple binary network coding at a subset of the network edges.

The paper is organized as follows. We formulate the problem in *Section II* and give the characterization of the rate region $\mathcal{R}_G^\alpha$ for a combination network G, in the presence of erasures of rate α in *Section III*. In *Section IV* we introduce an algorithm that uses topological information to achieve any rate pair $(R_1, R_2) \in \mathcal{R}_G^\alpha$. *Section V* shows an analytical comparison between our algorithm and a network coding approach, where the resources are allocated without any knowledge of the topological information. We conclude with some final remarks and directions for future work in *Section VI*. For the rest of the paper, we use the terms "edge" and "resource" interchangeably.

II. Problem formulation

The problem of interest is communication of a public message W_1 and a private message W_2 at rates R_1, and R_2 respectively, to a set of three receivers, $\mathcal{U} = \{1, 2, 3\}$. The transmission is performed over a combination network

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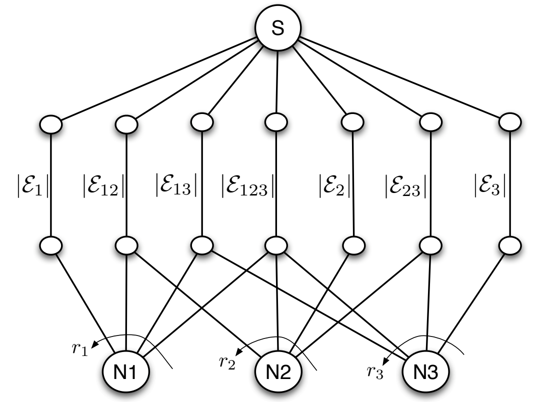

Fig. 1. Combination network with one source and three receivers. For clarity, we represent every set $\mathcal{E}_{\{.\}}$ using only one edge, and indicate set cardinality on the left side of that edge. Each receiver i has access to r_i edges.

G, illustrated in *Fig. 1*, where each channel has an erasure probability α and each receiver i has access to r_i edges. Message W_1 is required at all destinations, while message W_2 is only required at one of them, say the third receiver. Under this scenario, we set out to characterize the rate region $\mathcal{R}_G^\alpha$ at which messages W_1 and W_2 can be reliably communicated to the three receivers.

In this paper, we let $\mathcal{E}$ denote the total set of the intermediate edges, and $\mathcal{E}_i \subseteq \mathcal{E}$ denotes the set of the edges visible only to receiver i. Similarly $\mathcal{E}_{ij} \subseteq \mathcal{E}$ contains the edges visible only to receivers i and j and $\mathcal{E}_{ijk}$ is the set of edges visible to all three of the receivers. With this notation, we have that: $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_{12} \cup \mathcal{E}_{13} \cup \mathcal{E}_{23} \cup \mathcal{E}_{123}$, where each edge $e \in \mathcal{E}$ is visible to at least one receiver and it belongs to exactly one of the defined subsets.

Finally, we assume the size of the field over which the coding operations are performed is large enough, such that the linear combinations sent over the outgoing edges, if chosen randomly, are independent with high probability. Thus, the number of linear independent combinations received by each destination i is equal to r_i, the min-cut to each destination, and it is given by: $r_i = |\mathcal{E}_i| + \sum_{j \in \mathcal{U}, j \neq i} |\mathcal{E}_{ij}| + |\mathcal{E}_{123}|$. In particular,

$$r_1 = |\mathcal{E}_1| + |\mathcal{E}_{12}| + |\mathcal{E}_{13}| + |\mathcal{E}_{123}| \tag{1}$$
$$r_2 = |\mathcal{E}_2| + |\mathcal{E}_{12}| + |\mathcal{E}_{23}| + |\mathcal{E}_{123}| \tag{2}$$
$$r_3 = |\mathcal{E}_3| + |\mathcal{E}_{13}| + |\mathcal{E}_{23}| + |\mathcal{E}_{123}| \tag{3}$$

We also denote with r_{ij} the size of the union of the edges that two destinations i and j, $i \neq j$, observe. The received signal at receiver i is given by $\bar{Y}_i = [y_{i,1} \cdots y_{i,r_i}]^t$ where $y_{i,j}$ is the signal received on the j^{th} incoming edge of destination i. By $\bar{Y}_i^n = [y_{i,1}^n \cdots y_{i,r_i}^n]^t$ we denote the received signals at receiver i during a block length n.

III. Main result

In this paper, we characterize the capacity region of the degraded two message set scenario over a combination network with three receivers. We also propose a polynomial time algorithm which gives the encoding scheme to achieve any rate pair (R_1, R_2) in that rate region.

Theorem 1: Any achievable rate pair (R_1, R_2) in the degraded two message set scenario, applied over a combination network G with channels of independent erasure probability α lies in the region R_G^α characterized by

$$\begin{aligned} R_1 &\leq (1-\alpha)\min\{r_i\} && (4)\\ R_1 + R_2 &\leq (1-\alpha) r_3 && (5)\\ 2R_1 + R_2 &\leq (1-\alpha)(r_1 + r_2 + |\mathcal{E}_3|) && (6) \end{aligned}$$

Theorem 2: Any rate pair $(R_1, R_2) \in \mathcal{R}_G^\alpha$ is achievable using the encoding scheme proposed by *Algorithm 2*.
We give the proof to Theorem 1 in this section and prove Theorem 2 in *Section IV*.

A. Proof of Theorem 1

We prove here that $\mathcal{R}_G^\alpha$ characterizes an upper bound to R_1 and R_2:

$$\begin{aligned} nR_1 &\leq H(W_1) \\ &\leq H(W_1) - H(W_1|\bar{Y}_i^n) + H(W_1|\bar{Y}_i^n) \\ &\leq H(W_1) - H(W_1|\bar{Y}_i^n) + n\epsilon_i && (7)\\ &\leq I(\bar{Y}_i^n; W_1) + n\epsilon_i \\ &= H(\bar{Y}_i^n) - H(\bar{Y}_i^n|W_1) + n\epsilon_i && (8)\\ &\leq H(\bar{Y}_i^n) - H(\bar{Y}_i^n|X^n) + n\epsilon_i && (9)\\ &\leq \sum_{l=1}^{r_i} I(Y_{i,l}^n; X^n) + n\epsilon_i \\ &\leq nr_i(1-\alpha) + n\epsilon_i, && (10) \end{aligned}$$

where $\bar{Y}_i^n$ is the vector of received signals at receiver i. To obtain (7), we have used Fano's inequality (for any $\epsilon_i > 0$), and to obtain (9), we have used the fact that $W_1 \rightarrow X^n \rightarrow \bar{Y}_i^n$ forms a Markov chain.

We furthermore get from (8) that $H(\bar{Y}_i^n|W_1) \leq H(\bar{Y}_i^n) - nR_1 + n\epsilon_i$, and conclude that

$$\begin{aligned} I(\bar{Y}_i^n; X^n|W_1) &= H(\bar{Y}_i^n|W_1) - H(\bar{Y}_i^n|X^n) \\ &\leq I(\bar{Y}_i^n; X^n) - nR_1 + n\epsilon_i. && (11) \end{aligned}$$

Similarly, for any $\epsilon' > 0$,

$$\begin{aligned} n(R_1+R_2) &\leq H(W_1W_2) \\ &\leq H(W_1W_2) - H(W_1W_2|\bar{Y}_3^n) + n\epsilon_3 \\ &\leq I(\bar{Y}_3^n; W_1W_2) + n\epsilon_i \\ &= H(\bar{Y}_3^n) - H(\bar{Y}_3^n|X^n) + n\epsilon_3 \\ &\leq \sum_{l=1}^{r_3} I(Y_{3,l}^n; X^n) + n\epsilon_3 \\ &\leq nr_3(1-\alpha) + n\epsilon_3. && (12) \end{aligned}$$

Finally for any $\epsilon > 0$, we have

$$\begin{aligned} nR_2 &\leq H(W_2|W_1) \\ &\leq H(W_2|W_1) - H(W_2|\bar{Y}_3^n W_1) + n\epsilon \\ &= I(\bar{Y}_3^n; W_2|W_1) + n\epsilon \\ &\overset{(a)}{\leq} I(\bar{Y}_1^n \bar{Y}_2^n \bar{Y}_3^n; X^n|W_1) + n\epsilon \end{aligned}$$

$$
\begin{aligned}
&\leq \quad I(\bar{Y}_1^n; X^n|W_1) + I(\bar{Y}_2^n; X^n|\bar{Y}_1^n) \\
&\qquad + I(\bar{Y}_3^n; X^n|\bar{Y}_1^n\bar{Y}_2^n) + n\epsilon \\
&\overset{(b)}{\leq} \quad I(\bar{Y}_1^n; X^n) - nR_1 + n\epsilon_1 + I(\bar{Y}_2^n; X^n) - nR_1 \\
&\qquad + n\epsilon_2 + H(\bar{Y}_3^n|\bar{Y}_1^n, \bar{Y}_2^n) - H(\bar{Y}_3^n|X^n) + n\epsilon \\
&\leq \quad I(\bar{Y}_1^n; X^n) - nR_1 + n\epsilon_1 + I(\bar{Y}_2^n; X^n) - nR_1 \\
&\qquad + n\epsilon_2 + H(\bar{Y}_{\mathcal{E}_3}^n) - H(\bar{Y}_{\mathcal{E}_3}^n|X^n) + n\epsilon \\
&\overset{(c)}{\leq} \quad n(1-\alpha)\left(r_1 + r_2 + |\mathcal{E}_3|\right) - 2nR_1 + n\delta. \qquad (13)
\end{aligned}
$$

In the above chain of inequalities, (a) follows because $W_2 \to (W_1, X^n) \to \bar{Y}_3^n$ forms a Markov chain. To obtain (b), we first use the fact that $\bar{Y}_1^n \to (W_1, X^n) \to \bar{Y}_2^n$ forms a Markov chain and we then apply inequality (11) for $\epsilon_i = \epsilon' = \epsilon,\ i = 1, 2$. Finally, (c) follows because $I(\bar{Y}_{\mathcal{E}_3}^n; X^n) \leq n(1-\alpha)|\mathcal{E}_3|$.

B. Discussion

From the inequalities which characterize $\mathcal{R}_G^\alpha$, (4) and (5) are straightforward, as they essentially express min-cut conditions, while the third inequality and its effect on the rate region is more interesting, and we thus discuss it in more detail in the following.

Assume for simplicity that $\alpha = 0$, what intuitively the third inequality says is that if the r_1 edges to the first destination do not sufficiently overlap with the r_2 edges to the second destination, we may need to use twice the bottleneck edges in the combination network (hence the factor of 2) for W_1 to reach both these receivers. Then the rate R_2 we can send to the third receiver is limited by the "leftover" edges,

$$R_2 \quad \leq \quad (r_1 - R_1) + (r_2 - R_1) + |\mathcal{E}_3|, \qquad (14)$$

i.e. the edges that only the third receiver sees, and the edges remaining after duplicating message W_1 at rate R_1 to reach the first two receivers.

More formally, depending on the parameters of the topology, i.e. the number of edges in each set $\mathcal{E}_{\{.\}}$, the third inequality becomes active only for those topologies where the following situation occurs:

$$\min\{r_1, r_2, r_3\} + r_3 \quad > \quad r_1 + r_2 + |\mathcal{E}_3|. \qquad (15)$$

Note that if $r_3 = \min\{r_1, r_2, r_3\}$, then the above relation does not hold, since $r_i \geq r_3, i \in \{1, 2\}$. Therefore, r_3 does not affect the value of $\min\{r_1, r_2, r_3\}$ and we equivalently have the third inequality active when

$$\min\{r_1, r_2\} + r_3 \quad > \quad r_1 + r_2 + |\mathcal{E}_3|. \qquad (16)$$

Replacing the corresponding values of the ranks, we obtain that:

$$\min\{|\mathcal{E}_1| + |\mathcal{E}_{13}|, |\mathcal{E}_2| + |\mathcal{E}_{23}|\} > |\mathcal{E}_1| + |\mathcal{E}_2| + |\mathcal{E}_{12}|. \qquad (17)$$

Fig. 2 is an example of such topological parameters. We give in the following an algorithm to verify for a desired combination network if the third inequality becomes active.

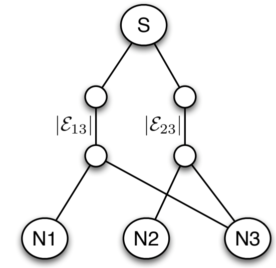

Fig. 2. Canonical combination network for the case when inequality (6) is active. After running *Algorithm 1* on a given combination network, only sets $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$ still contain edges.

The proof can be found in [10]. It turns out that *Fig. 2* is the canonical combination network with the third inequality active; i.e. *Algorithm 1* returns ACTIVE if and only if *Fig. 2* is the combination network that remains after the edge eliminations up to that iteration.

Algorithm 1 This algorithm returns ACTIVE when the third inequality is active depending on the topological parameters and returns NOT ACTIVE otherwise.

```
while TRUE do
    if |E13| = 0 OR |E23| = 0 then
        return NOT ACTIVE
    end if
    if |E3| > 0 then
        |E3| ← |E3| − 1
    else if |E12| > 0 AND |E13| > 0 AND |E23| > 0 then
        |E13| ← |E13| − 1; |E23| ← |E23| − 1; |E12| ← |E12| − 1
    else if |E13| > 0 AND |E2| > 0 then
        |E13| ← |E13| − 1; |E2| ← |E2| − 1
    else if |E23| > 0 AND |E1| > 0 then
        |E23| ← |E23| − 1; |E1| ← |E1| − 1
    else
        return ACTIVE
    end if
end while
```

IV. Algorithm Description

In this section we introduce an algorithm that uses topological information in order to achieve any desired rate pair $(R_1, R_2) \in \mathcal{R}_G^\alpha$. The algorithm uses the fact that each intermediate edge is essentially one available resource to the set of receivers that are connected to it and can carry linear combinations of W_1 and W_2. We show that we do not need to perform network coding among W_1 and W_2 in order to have an optimal algorithm (our *Algorithm 2* is such an example). For the sake of simplicity we consider the case of no erasures in *Section IV-A* and give the sketch of the proof for the case where each channel has an independent and uniform erasure probability of α in *Section IV-B*.

The idea of the algorithm is that the source puts linear combinations of symbols of W_1 or of W_2 on each of the edges so that it guarantees decodability of W_1 at all the receivers and decodability of W_2 at the third receiver. We are interested in assigning each resource to carry one of the two messages. We indicate this by coloring the intermediate edges with two colors, t_1 for W_1 and t_2 for W_2, where $t_1 \neq t_2$. This edge assignment (edge coloring) is the output of our proposed *Algorithm 2* for a given rate pair $(R_1, R_2) \in \mathcal{R}_G^0$. The algorithm makes use of two methods, which we explain briefly. Function *FindEdge($\mathcal{A}$)* returns true if the set $\mathcal{A}$ contains at least an edge that has not been assigned for any message yet. Function *ColorEdge($\mathcal{A}, t_i$)* marks an edge of the specified set $\mathcal{A}$ to carry message W_i.

Algorithm 2 This algorithm assigns either t_1 or t_2 to each of the available resources, for a given rate pair $(R_1, R_2) \in \mathcal{R}_G^\alpha$.

1: **Input**: $(R_1, R_2) \in \mathcal{R}_G^\alpha$
2: **Initialize**: $c_e \leftarrow 0,\ \forall e \in \mathcal{E}$
3: **while** $R_1 > 0$ **do**
4:   **if** FindEdge($\mathcal{E}_{123}$) **then**
5:     $R_1 \leftarrow R_1 - 1$
6:     ColorEdge($\mathcal{E}_{123}, t_1$)
7:   **else if** FindEdge($\mathcal{E}_{13}$) AND FindEdge($\mathcal{E}_{23}$) AND FindEdge($\mathcal{E}_{12}$) **then**
8:     **if** $R_1 \geq 2$ **then**
9:       $R_1 \leftarrow R_1 - 2$
10:       ColorEdge($\mathcal{E}_{13}, t_1$);ColorEdge($\mathcal{E}_{23}, t_1$);ColorEdge($\mathcal{E}_{12}, t_1$)
11:     **else**
12:       $R_1 \leftarrow R_1 - 1$
13:       ColorEdge($\mathcal{E}_{13}, t_1$); ColorEdge($\mathcal{E}_{12}, t_1$)
14:     **end if**
15:   **else if** FindEdge($\mathcal{E}_i$) AND FindEdge($\mathcal{E}_{jk}$), $\{i,j,k\} = \{1,2,3\}$ **then**
16:     $R_1 \leftarrow R_1 - 1$
17:     ColorEdge($\mathcal{E}_i, t_1$); ColorEdge($\mathcal{E}_{jk}, t_1$)
18:   **else if** FindEdge($\mathcal{E}_1$) AND FindEdge($\mathcal{E}_2$) AND FindEdge($\mathcal{E}_3$) **then**
19:     $R_1 \leftarrow R_1 - 1$
20:     ColorEdge($\mathcal{E}_1, t_1$); ColorEdge($\mathcal{E}_2, t_1$); ColorEdge($\mathcal{E}_3, t_1$)
21:   **else if** FindEdge($\mathcal{E}_{ij}$) AND FindEdge($\mathcal{E}_{ik}$), $\{i,j,k\} = \{1,2,3\}$ **then**
22:     $R_1 \leftarrow R_1 - 1$
23:     ColorEdge($\mathcal{E}_{ij}, t_1$); ColorEdge($\mathcal{E}_{ik}, t_1$)
24:   **end if**
25: **end while**
26: Assign R_2 edges from the remaining edges visible to receiver 3 to carry W_2

One should note that network coding is actually needed only for step 7 of *Algorithm 2*, when it assigns resources from the sets visible to all two receivers, $\mathcal{E}_{ij}$. By selecting an edge from each $\mathcal{E}_{ij}$, and sending a linear combination of W_1 on each of them, every destination receives a total rate of two. For the remaining situations, it is enough to route by conveniently selecting one edge from those sets that complement each other, for example sets $\mathcal{E}_2$ and $\mathcal{E}_{13}$ as long as the sets still contain edges that have not been assigned yet.

A. Algorithm optimality - no erasures

Lemma 1: Algorithm 2 stops after finite steps.

Proof: We first prove that after each iteration (inside the while loop) R_1 is decreased by at least 1. We then conclude that *Algorithm 2* stops after at most R_1 iterations. In each iteration, R_1 is decreased if either of the "IF conditions" are satisfied. No "IF condition" is satisfied only when all $|\mathcal{E}_{123}|$, $\min\{|\mathcal{E}_{12}|, |\mathcal{E}_{13}|, |\mathcal{E}_{23}|\}$, $\min\{|\mathcal{E}_i|, |\mathcal{E}_{j,l}|\}$, $\min\{|\mathcal{E}_1|, |\mathcal{E}_2|, |\mathcal{E}_3|\}$ and $\min\{|\mathcal{E}_{i',j'}|, |\mathcal{E}_{i',l'}|\}$ are already assigned which ensures R_1 having been decreased by at least r_1 or r_2. But then since $(R_1, R_2) \in \mathcal{R}_G^0$, R_1 should satisfy $R_1 \leq \min\{r_1, r_2\}$ which means that $R_1 \leq 0$ in the studied iteration and this is a contradiction. ∎

Lemma 2: Algorithm 2 is optimal

Proof: We prove here that for any $(R_1, R_2) \in \mathcal{R}_G^0$, the assignment proposed by *Algorithm 2* lets (i) all receivers get R_1 random linear combinations of W_1 and (ii) the third receiver further gets R_2 random linear combinations of W_2. The proof is by induction:

Induction Base: Let $R_1 = 0$. *Algorithm 2* assigns W_2 to all the resources. Thus, receiver 3 gets $r_3 \geq R_1 + R_2$ random linear combinations of W_2 and (i) and (ii) both hold.

Induction Hypothesis: Let $R_1 \leq r$ and assume that the assignment given by *Algorithm 2* satifies (i) and (ii) for any $(R_1, R_2) \in \mathcal{R}_G^0$ and over all combination networks.

Induction Step: Assume $R_1 = r + 1$ and $(R_1, R_2) \in \mathcal{R}_G^0$. Run *Algorithm 2* for one iteration to assign message W_1 on the edge(s) $\mathbf{e}$ that it finds, providing each receiver k, $k = 1, 2, 3$ with $r_k^{\mathbf{e}} \geq 1$ linear combinations of W_1. We show that eliminating these edges leaves us with a combination network G' on which resources could be allocated to $(R_1 - \min_k\{r_k^{\mathbf{e}}\})$ rate of message W_1 and R_2 rate of message W_2. To this end, we show that $(R_1 - \min_k\{r_k^{\mathbf{e}}\}, R_2) \in \mathcal{R}_{G'}^0$, where $R_1 - \min_k\{r_k^{\mathbf{e}}\} \leq r$ and $\mathcal{R}_{G'}^0$ is the capacity region of the new combination network G'. We then apply the induction hypothesis (which states that *Algorithm 2* optimally gives the resource assignment on G' for all rate pairs $(R_1', R_2') \in R_{G'}^0$, $R_1' \leq r$) to conclude the optimality of *Algorithm 2*.

We take into account the following cases as suggested by *Algorithm 2* and find the structure of G' which is formed after the edge elimination depending on the topology of the combination network.

- $|\mathcal{E}_{123}| > 0$. The edge to be marked in this case is an edge of $\mathcal{E}_{123}$. It is easy to see that $\min_k\{r_k^{\mathbf{e}}\} = 1$ and the resulting G' has $r_k' = r_k - 1,\ k = 1, 2, 3$, and $|\mathcal{E}_3'| = |\mathcal{E}_3|$.
- $|\mathcal{E}_{123}| = 0$, and $\min\{|\mathcal{E}_{12}|, |\mathcal{E}_{13}|, |\mathcal{E}_{23}|\} > 0$. In this case, one edge from each $\mathcal{E}_{ij}$ is marked. We thus have $\min_k\{r_k^{\mathbf{e}}\} = 2$ and G', depending on R_1, has either $r_k' = r_k - 2,\ k = 1, 2, 3$, and $|\mathcal{E}_3'| = |\mathcal{E}_3|$ (if $R_1 \geq 1$) or $r_1' = r_1 - 2$, $r_2' = r_2 - 1$, $r_3' = r_3 - 1$, and $|\mathcal{E}_3'| = |\mathcal{E}_3|$ (if $R_1 = 1$).
- $|\mathcal{E}_{123}| = \min\{|\mathcal{E}_{12}|, |\mathcal{E}_{13}|, |\mathcal{E}_{23}|\} = 0$, and $|\mathcal{E}_i| \& |\mathcal{E}_{j,l}| > 0$ for some $\{i, j, l\} = \{1, 2, 3\}$. In this case, one edge from $\mathcal{E}_i$ and one edge from $\mathcal{E}_{j,l}$ is marked. So $\min_k\{r_k^{\mathbf{e}}\} = 1$ and G' has the following topological parameters: $r_k' = r_k - 1,\ k = 1, 2, 3$, and either $|\mathcal{E}_3'| = |\mathcal{E}_3|$ (if $i \neq 3$) or $|\mathcal{E}_3'| = |\mathcal{E}_3| - 1$ (if $i = 3$).
- $|\mathcal{E}_{123}| = \min\{|\mathcal{E}_{12}|, |\mathcal{E}_{13}|, |\mathcal{E}_{23}|\} = \min\{|\mathcal{E}_i|, |\mathcal{E}_{j,l}|\} = 0,\ \forall\{i, j, l\} = \{1, 2, 3\}$, and $|\mathcal{E}_1| \& |\mathcal{E}_2| \& |\mathcal{E}_3| > 0$. In

this case, one edge from each $\mathcal{E}_i$ is marked. Similarly, $\min_k\{r_k^{\mathbf{e}}\} = 1$ and G' has $r'_k = r_k - 1,\ k = 1, 2, 3$, and $|\mathcal{E}'_3| = |\mathcal{E}_3| - 1$.

- $|\mathcal{E}_{123}| = \min\{|\mathcal{E}_{12}|, |\mathcal{E}_{13}|, |\mathcal{E}_{23}|\} = \min\{|\mathcal{E}_i|, |\mathcal{E}_{j,l}|\} = \min\{|\mathcal{E}_1|, |\mathcal{E}_2|, |\mathcal{E}_3|\} = 0,\ \forall\{i, j, l\} = \{1, 2, 3\}$, and $|\mathcal{E}_{i,j}|\&|\mathcal{E}_{i,l}| > 0$ for some $\{i, j, l\} = \{1, 2, 3\}$. In this case, we have one edge from $\mathcal{E}_{ij}$ and one edge from $\mathcal{E}_{il}$ marked. $\min_k\{r_k^{\mathbf{e}}\} = 1$ and G' has $r'_i = r_i - 2$, $r'_j = r_j - 2$, $r'_l = r_l - 2$ and $|\mathcal{E}'_3| = |\mathcal{E}_3|$.

For all those cases with $r'_k = r_k - 1,\ k = 1, 2, 3$, and $|\mathcal{E}'_3| = |\mathcal{E}_3| - 1$, $\mathcal{R}^0_{G'}$ is characterized by

$$\begin{aligned} R'_1 &\leq \min\{r_1, r_2, r_3\} - 1, && (18)\\ R'_1 + R'_2 &\leq r_3 - 1, && (19)\\ 2R'_1 + R'_2 &\leq r_1 - 1 + r_2 - 1 + |\mathcal{E}_3|. && (20)\end{aligned}$$

Furthermore, in all such cases, $\min_k r_k^{\mathbf{e}} = 1$ and so it's easy to verify that $(R_1 - \min_k r_k^{\mathbf{e}}, R_2) \in \mathcal{R}^0_{G'}$ for all $(R_1 = r + 1, R_2) \in \mathcal{R}^0_G$. The same argument should be made for all the other cases. For the sake of brevity we present here the case where $|\mathcal{E}_{123}| = 0$ and $\min\{|\mathcal{E}_{12}|, \mathcal{E}_{13}|, |\mathcal{E}_{23}|\} > 0$ (which is interestingly the only case where routing is not optimal). We consider two cases: $R_1 \geq 2$ and $R_1 = 1$.

- $R_1 \geq 2$: $\mathcal{R}^0_{G'}$ is characterized by

$$\begin{aligned} R'_1 &\leq \min\{r_1, r_2, r_3\} - 2, && (21)\\ R'_1 + R'_2 &\leq r_3 - 2, && (22)\\ 2R'_1 + R'_2 &\leq r_1 - 2 + r_2 - 2 + |\mathcal{E}_3|. && (23)\end{aligned}$$

  Furthermore, $\min_k r_k^{\mathbf{e}} = 2$. It is immediate to see that $(R_1 - \min_k r_k^{\mathbf{e}}, R_2) \in \mathcal{R}^0_{G'}$ for all $(R_1 = r+1 > 1, R_2) \in \mathcal{R}^0_G$.

- $R_1 = 1$: $\mathcal{R}^0_{G'}$ is characterized by

$$\begin{aligned} R'_1 &\leq \min\{r_1 - 2, r_2 - 1\}, && (24)\\ R'_1 + R'_2 &\leq r_3 - 1, && (25)\\ 2R'_1 + R'_2 &\leq r_1 - 2 + r_2 - 1 + |\mathcal{E}_3|. && (26)\end{aligned}$$

  Furthermore, $\min_k r_k^{\mathbf{e}} = 1$. We prove by contradiction that for all $(R_1 = 1, R_2) \in \mathcal{R}^0_G$, we have $(R_1 - \min_k r_k^{\mathbf{e}} = 0, R_2) \in \mathcal{R}^0_{G'}$. Assume that $(0, R_2) \notin \mathcal{R}^0_{G'}$ for some R_2 which satisfies $(1, R_2) \in \mathcal{R}^0_G$. Then

$$\min\begin{Bmatrix} r_3 - 1, \\ r_1{+}r_2{+}|\mathcal{E}_3|{-}3 \end{Bmatrix} < \min\begin{Bmatrix} r_3 - 1, \\ r_1{+}r_2{+}|\mathcal{E}_3|{-}2 \end{Bmatrix}. \quad (27)$$

  We show in the following that to have (27), we should have $r_1+r_2-3+|\mathcal{E}_3| < r_3-1 < r_1+r_2-2+|\mathcal{E}_3|$ which is a contradiction (for our assumed integer values): The right hand side can be simplified to r_3-1 and furthermore

$$\begin{aligned} r_3 - 1 &\overset{(1)}{=} |\mathcal{E}_3| + |\mathcal{E}_{13}| + |\mathcal{E}_{23}| - 1 && (28)\\ &\leq |\mathcal{E}_3| + |\mathcal{E}_{13}| + |\mathcal{E}_{23}| + |\mathcal{E}_1| + \\ &\quad + |\mathcal{E}_2| + 2(|\mathcal{E}_{12}| - 1) - 1 && (29)\\ &\overset{(2)}{=} r_1 - 1 + r_2 - 3 + |\mathcal{E}_3| && (30)\\ &< r_1 - 1 + r_2 - 2 + |\mathcal{E}_3|, && (31)\end{aligned}$$

  where (1) and (2) are both by the assumption of $|\mathcal{E}_{123}| = 0$. The left hand side is thus not equal to $r_3 - 1$, forcing $r_1 - 1 + r_2 - 3 + |\mathcal{E}_3| < r_3 - 1 < r_1 - 1 + r_2 - 2 + |\mathcal{E}_3|$: contradiction. So $(R_1 - \min_k r_k^{\mathbf{e}} = 0, R_2) \in \mathcal{R}^0_{G'}$ for all $(R_1 = 1, R_2) \in \mathcal{R}^0_G$.

The reader is refered to [10] for the analysis of $(R_1 - \min_k r_k^{\mathbf{e}}, R_2) \in \mathcal{R}^0_{G'}$ in the other cases. ■

B. Algorithm optimality - erasures

In this section, we assume an erasure probability $\alpha > 0$ for all the channels of the combination network independently. To communicate messages W_1 and W_2 of rates $(R_1, R_2) \in \mathcal{R}^\alpha_G$, we use a random code of rate $(1 - \alpha)$ and encode the nR_1 symbols of W_1 to $\frac{nR_1}{1-\alpha}$ symbols and similarly symbols of W_2 to $\frac{nR_2}{1-\alpha}$ symbols. Linear combinations of encoded W_1 and also of encoded W_2 symbols are now of a rate smaller than $1 - \alpha$ and can be communicated to the intermediate nodes with arbitrary small error probability. We can thus apply *Algorithm 2* for the rate pair $(\frac{nR_1}{1-\alpha}, \frac{nR_2}{1-\alpha})$ and we just have to guarantee that messages $\hat{W}_1$ and $\hat{W}_2$ could be re-constructed such that

$$Pr\{\hat{W}_i \neq W_i\} \overset{n\to\infty}{\to} 0. \quad (32)$$

Since the receivers are provided with random linear combinations of encoded message W_1 and random linear combinations of encoded message W_2, (32) holds if the following two conditions are satisfied with high probability:

- The number of non-erased W_1 carrying signals received at each receiver is greater than or equal to nR_1 with high probability, and
- The number of non-erased W_2 carrying signals received at receiver 3 is greater than or equal to nR_2 with high probability.

Consider the received vector $\bar{Y}_i^n$ at receiver i. By the algorithm analysis in *Section IV-A*, we know that each receiver i is connected to at least $\frac{nR_1}{1-\alpha}$ edges which carry linear combinations of the randomly encoded W_1 (with high probability). Pick the set (of cardinality $\frac{nR_1}{1-\alpha}$) of those edges carrying the aforementioned $\frac{nR_1}{1-\alpha}$ linear combinations. By some abuse of notation, call them $Y_1, \cdots, Y_{\frac{nR_1}{1-\alpha}}$. Assign to each Y_k a random variable Z_k defined as

$$Z_k = \begin{cases} 0 & \text{if } Y_k \text{ is erased} \\ 1 & \text{otherwise} \end{cases}. \quad (33)$$

Since $Pr\{|\sum_k Z_k - \frac{nR_1}{1-\alpha} \times (1-\alpha)| \geq \epsilon\} \to 0$ when $n \to \infty$, the number of non-erased W_1-carrying signals received at each receiver is greater than or equal to nR_1 with high probability. Similarly for W_2. This concludes the achievability of the rate pair $(R_1, R_2) \in \mathcal{R}^\alpha_G$.

V. Algorithm Evaluation

In this section we compare the encoding scheme given by *Algorithm 2* described in previous sections, with a network coding-based scheme which we denote by *NCrand*.

For the *NCrand* scheme, the source has only information about the min-cut of each receiver, and it does not know which edge is available to what receiver. The server uses all the available resources and for each message $W_k, k \in \{1,2\}$ it randomly allocates a number of edges, proportional to the rate R_k that should be delivered. This means that for any rate pair $(R_1,R_2) \in \mathcal{R}_G^0$, during each time slot, the server selects randomly $\frac{R_1}{R_1+R_2} r_{123}$ edges to send W_1 and $\frac{R_2}{R_1+R_2} r_{123}$ edges to send W_2. Then, each destination i receives $r_i \frac{R_1}{R_1+R_2}$ linear combinations of W_1, but at most R_1 linear combinations are linearly independent. Therefore, the useful rate of W_1 at receiver i with $i \in \{1,2,3\}$ is given by:

$$S_{1,i} = R_1 \min\left\{1, \frac{r_i}{R_1+R_2}\right\}. \tag{34}$$

Analogously, the useful rate of message W_2 at the third receiver is equal to:

$$S_{2,3} = R_2 \min\left\{1, \frac{r_3}{R_1+R_2}\right\} = R_2, \tag{35}$$

using inequality (5) from the characterization of the rate region. Notice that $S_{2,1}$ and $S_{2,2}$ are not of interest, since only the third receiver should receive message W_2.

Consider we use the network during T time slots. For any rate pair $(R_1,R_2) \in \mathcal{R}_G^0$, the *Algorithm 2* delivers a total rate of $T(R_1+R_2)$, as in each time slot it is able to assign the resources such that to achieve the desired rate pair. In order to deliver the same total rate with *NCrand*, the server needs T_r time slots, where $T_r = \max\{T_1, T_2\}$. Further, T_1 is the total number of time slots needed to deliver message W_1 to all receivers:

$$T_1 = \max\left\{\frac{TR_1}{S_{1,1}}, \frac{TR_1}{S_{1,2}}, \frac{TR_1}{S_{1,3}}\right\} = \frac{T}{\min\left\{1, \frac{\min_i r_i}{R_1+R_2}\right\}} \tag{36}$$

T_2 is the number of time slots needed to deliver message W_2 to the third receiver:

$$T_2 = \frac{TR_2}{S_{2,3}} = T. \tag{37}$$

Next, we define the following function to measure the benefit of using *Algorithm 2* over the *NCrand* scheme:

$$\begin{aligned} f(R_1,R_2) &= \frac{T_r}{T} = \frac{\max\{T_1,T_2\}}{T} & (38)\\ &= \max\left\{\frac{1}{\min\left\{1, \frac{\min_i\{r_i\}}{R_1+R_2}\right\}}, 1\right\} & (39) \end{aligned}$$

for any rate pair $(R_1,R_2) \in \mathcal{R}_G^0$. If $f(R_1,R_2)$ takes higher values, this means the time needed to deliver a desired rate is shorter for the scheme proposed by *Algorithm 2* as compared to the *NCrand* approach.

Algorithm 2 provides benefits over the other approach if $f(R_1,R_2) > 1$, which occurs for the case when $R_1+R_2 > \min_i\{r_i\}$. Note that in this situation, the bottleneck is either receiver 1 or receiver 2, since $r_3 \geq R_1+R_2$ from inequality (5) from the rate region. Intuitively, if we consider that receiver 1 has access to fewer resources than the others, with *NCrand* the server may select the resources visible to 1 to carry W_2. Consequently, the leftover edges to which receiver 1 has access, are not enough to deliver message W_1 to him in one time slot. If $R_1+R_2 \leq \min_i\{r_i\}$, then the *NCrand* delivers the desired rate pair per time slot, as our algorithm, and $f(R_1,R_2) = 1$.

For example, given the topology in *Fig. 2*, for a network with $|\mathcal{E}_{13}| = 3$ and $|\mathcal{E}_{23}| = 4$, we have that $r_1 = 3$, $r_2 = 4$, and $r_3 = 7$. In order to deliver rate pair $(R_1,R_2) = (3,1)$, *Algorithm 2* outperforms the *NCrand* approach by 33%, with $f(3,1) = \frac{4}{3}$. However, for rate pair $(R_1,R_2) = (1,1)$, both schemes use the same number of time slots, and $f(1,1) = 1$.

VI. Conclusions and future work

In this paper we studied the degraded two message set problem, over a combination network G and in the presence of erasures of rate α. We gave a characterization of the rate region, $\mathcal{R}_G^\alpha$, and introduced an algorithm that achieves it by using topological information. Further we compared our algorithm to an approach oblivious to the network topology that selects the resources at random, and found out that the benefits obtained with the proposed algorithm depends both on the available resources and the rate pair that we want to achieve. In particular, relying on the knowledge about the network topology, the server can deliver messages W_1 and W_2 even at the highest rates from the rate region, using *Algorithm 2*. Without topological knowledge, the server can only achieve low rate pairs.

As future work, we consider extending the algorithm to the case of multicasting to a larger set of receivers and carry on a practical evaluation.

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